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Chess-like Games Are Dominance Solvable in at Most Two Steps

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We show that strictly competitive, finite games of perfect information that may end in one of three possible ways can be solved by applying only two rounds of elimination of dominated strategies. *Journal of Economic Literature* Classification Number: C72. © 2000 Academic Press

1. INTRODUCTION

During the pioneering time of game-theoretic research in the first half of the past century, it was proved that the game of chess has a value, i.e., that either there is a winning strategy for White, or there is a winning strategy for Black, or both players can secure themselves a draw. Since then, a rich theory has developed for general games of perfect information,[2] yet surprisingly little is implied by this theory for specific games such as chess; e.g., the question as to which of the three possible values of chess is the actual value continues to be unsettled.

In this paper we show that *chess-like games*, i.e., strictly competitive, finite games of perfect information with at most three outcomes, can be solved by applying only *two* rounds of elimination of dominated strategies in the strategic form. In plain words, our result says that these games (one of which is chess) have the following property. Consider the strategic form of the game. Eliminate all dominated strategies from the strategy sets of each

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[2]While von Neumann and Morgenstern (1944) developed the notion of an extensive-form game first, the by now standard formulation of extensive-form (and in particular, of perfect-information) games was given by Kuhn (1953). For an introduction to the theory of games of perfect information, see Osborne and Rubinstein (1994, Chapter II) and references therein.

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player. Repeat this procedure once. Then the remaining strategy profiles all induce the same outcome, which corresponds to the value of the game.

An analogous result for games with at most two outcomes is easily established: Fix any strictly competitive, finite game of perfect information with at most two outcomes.[3] Then, by the minmax theorem, one of the players, say player i, has a winning strategy. Clearly, this strategy dominates all of player i's non-winning strategies, so that a single round of elimination reduces the strategy set of player i to the set of winning strategies. Thus, any strictly competitive, finite game of perfect information with at most two outcomes is dominance solvable in (at most) one step.

2. NOTATION AND DEFINITIONS

We consider any finite, strictly competitive game G of perfect information with at most three outcomes.[4] Formally, let

$$G = (X, x^0, \alpha, \iota, \omega),$$

where X is a finite set of *nodes*, $x^0 \in X$ is the *initial node*, α: $X\backslash\{x^0\} \to X$ is the *anterior node function*, and if $Z = X\backslash\alpha(X)$ denotes the set of *terminal nodes*,[5] then ι: $X\backslash Z \to \{1, 2\}$ denotes the *player function*, and $\omega = (\omega_1, \omega_2)$: $Z \to \{(1, -1), (0, 0), (-1, 1)\}$ denotes the *outcome function*. For a given pair of nodes x, x' we say that x' *precedes* x when there is an $H \geq 0$ and a sequence $x' = x_0, x_1, \ldots, x_H = x$ such that $\alpha(x_h) = x_{h-1}$ for $h = 1, \ldots, H$. Let $X(x)$ denote the set of nodes preceded by x. Note that by definition, x is contained in $X(x)$. We require that x^0 precedes any $x \in X$. Let x^1 be any node. The *subgame* rooted in x^1 is defined by

$$G(x^1) := (X(x^1), x^1, \alpha^1, \iota|_{X(x^1)\backslash Z^1}, \omega|_{Z^1}),$$

where $\alpha^1 := \alpha|_{X(x^1)\backslash Z^1}$ and $Z^1 := X(x^1)\backslash\alpha^1(X(x^1))$.[6] For any non-terminal node $x \in X\backslash Z$, let $A_x := \{y \in X|\alpha(y) = x\}$ be the set of *actions* available at node x. By an *action profile* we mean a tuple $s = (a_x)_{x\in X\backslash Z}$, where $a_x \in A_x$. The terminal node $z(s) \in Z$ determined by s is characterized by the property that there exists a sequence $x^0 = x_0, x_1, \ldots, x_H = z(s)$ such that $a_{x_{h^*-1}} = x_h$ for $h = 1, \ldots, H$. Call $(x_0, x_1, \ldots, x_H)$ the *path* $p(s)$ leading to $z(s)$. If, for a given node x and an action profile s, we have

[3]These are games that cannot end in a draw. Examples include Nim, Hex, and other simple games played for diversion (cf. Binmore, 1992).

[4]All games will be understood to be without chance moves.

[5]We follow standard notation when applying functions to sets, e.g., $\alpha(X) := \{\alpha(x) \mid x \in X\}$.

[6]As usual, for a mapping f, the restriction of f to a subset Y of its domain is denoted by $f|_Y$.

$x = x_h$ for some index $h \in \{0, ..., H\}$, then we will say that s *reaches* $G(x)$ (or simply x). A finite *two-person game in normal form* is a quadruple

$$N = (S_1, S_2, u_1(.), u_2(.)),$$

where S_i is player i's (finite) *strategy set* and u_i: $S_1 \times S_2 \to \Re$ is player i's *utility function*. The *strategic form* of the perfect information game G is the normal-form game $N(G) := (S_1, S_2, u_1(.), u_2(.))$, with strategy sets

$$S_i := \prod_{\substack{x \in X \setminus Z \\ \iota(x)=i}} A_x,$$

and utility functions given by $u_i(s) := \omega_i(z(s_1, s_2))$, for $i = 1, 2$.[7] A strategy s_i is *dominated* by another strategy s_i', if $u_i(s_i, s_j) \leq u_i(s_i', s_j)$ for all s_j and there exists an s_j' such that $u_i(s_i, s_j') < u_i(s_i', s_j')$.[8] Strategy s_i is said to be dominated (without qualification) if it is dominated by some other strategy.[9] Let $N = (S_1, S_2, u_1(.), u_2(.))$ be a finite two-person game in normal form. Then the game resulting from N by elimination of dominated strategies is defined as

$$D(N) := (\tilde{S}_1, \tilde{S}_2, u_1|_{\tilde{S}_1 \times \tilde{S}_2}(.), u_2|_{\tilde{S}_1 \times \tilde{S}_2}(.)),$$

where $\tilde{S}_i$ is the set of strategies for player i that are not dominated in N. Let $D^1(N) := D(N)$ and $D^k(N) := D(D^{k-1}(N))$ for $k > 1$. We will say that $D^k(N)$ results from k-fold *elimination of dominated strategies*. A normal-form game N is called *dominance solvable*[10] in at most k steps if the utility functions in $D^k(N)$ are constant. Note that if N is dominance solvable in at most k steps then, for any $K \geq k$, $D^k(N) = D^K(N)$ and N is dominance solvable in at most K steps. The *value* of a normal-form game N for player i is

$$v_i(N) := \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j).$$

For a node x in a perfect information game G, we will write $v_i(x) := v_i(N(G(x)))$ for the value of the subgame rooted in x to player i. A strategy $s_i \in S_i$ is a *maximin strategy* if $s_i \in \arg\max_{\tilde{s}_i \in S_i} \min_{s_j \in S_j} u_i(\tilde{s}_i, s_j)$. We denote the set of maximin strategies for player i in N by $M_i(N)$. For future

[7]The strategic form is non-reduced in the sense that it may contain *equivalent* strategies, i.e., strategies s_i and s_i' such that $u_1(s_i, .) \equiv u_1(s_i', .)$ and $u_2(s_i, .) \equiv u_2(s_i', .)$.

[8]As usual, $j \in \{1, 2\}$, and $j \neq i$. Note that we abuse notation by changing the order of the arguments of the utility functions when $i = 2$.

[9]For a discussion of the form of dominance used here, see Börgers (1993).

[10]The notion of a "dominance solvable" game was introduced by Moulin (1979). See also Moulin (1986).

reference, we state the famous:

PROPOSITION 1 (minmax theorem). *Let N be the strategic form of a finite perfect-information game of conflicting interests. Then $v_1(N) = -v_2(N)$.*

Proof. See Binmore (1992, p. 44). ■

3. THE RESULT

THEOREM 1. *Let $N = N(G)$ be the strategic form of a finite, strictly competitive game of perfect information with at most three outcomes. Then N is dominance solvable in at most two steps.*

Proof. By the minmax theorem, either one of the players has a winning strategy[11] (Case A), or each of them can enforce a draw (Case B).

Case A. Without loss of generality, assume that player 1 has a winning strategy, i.e., $v_1(N) = 1$. Then, by definition, a strategy s_1 is a maximin strategy for player 1 if and only if

$$u_1(s_1, .) \equiv 1. \tag{1}$$

Therefore, a strategy of player 1 is dominated (by s_1) if and only if it is not a maximin strategy. Consequently, player 1's strategy set in $D(N)$ is $M_1(N)$. From (1) we see that the utility functions for both players are constant in $D(N)$ and hence that N is dominance solvable in (at most) one step. This proves the theorem for $v_i(N) = \pm 1$.

Case B. Let now $v_1(N) = v_2(N) = 0$. Assume that $u_1(s_1, s_2) = -1$ for some strategy profile (s_1, s_2), where both s_1 and s_2 are undominated in N. To prove the theorem, it suffices to show that s_1 is dominated in $D(N)$. By assumption, $N = N(G)$ for some finite game G of perfect information. Consider the path $p(s) = (x_0, x_1, \ldots, x_H)$ associated with the action profile $s = (s_1, s_2)$. By assumption, $v_1(x_0) = 0$ and $v_1(x_H) = -1$. Let x_{h^*} be the first node (i.e., with lowest index) on the path $p(s)$ with non-zero value of the subgame rooted in x_{h^*}.

We will show that $v_1(x_{h^*}) = -1$. This is clear if $h^* = H$. Therefore let $h^* < H$. To provoke a contradiction assume that $v_1(x_{h^*}) = 1$. Consider the alternative strategy $\tilde{s}_1 \in S_1$ which is equal to s_1 at all nodes $x \in X \backslash X(x_{h^*})$, i.e., outside of the subgame rooted in x_{h^*}, and which is equal to some winning strategy in $G(x_{h^*})$. We show that $\tilde{s}_1$ dominates s_1 in N. Clearly, $u_1(\tilde{s}_1, \tilde{s}_2) \geq u_1(s_1, \tilde{s}_2)$ for all $\tilde{s}_2 \in S_2$ (this holds obviously for all $\tilde{s}_2$ such

[11] By a winning strategy in G, we mean a maximin strategy in $N(G)$ guaranteeing a utility of 1 for the player in question.

that $(s_1, \tilde{s}_2)$ reaches x_{h^*}, and with equality for all $\tilde{s}_2$ such that $(s_1, \tilde{s}_2)$ does not reach x_{h^*}). Moreover, as $(\tilde{s}_1, s_2)$ reaches x_{h^*},

$$u_1(\tilde{s}_1, s_2) = 1 > -1 = u_1(s_1, s_2),$$

so that s_1 is dominated by $\tilde{s}_1$ in N, thereby contradicting the assumption that s_1 is a strategy in $D(N)$.

We have shown that $v_1(x_{h^*}) = -1$, and we know from the definition of h^* that $v_1(x_{h^*-1}) = 0$. Hence, by the minmax theorem, player 2 has a winning strategy in the subgame rooted in x_{h^*}, yet no winning strategy in the subgame rooted in x_{h^*-1}. Consequently, at node x_{h^*-1}, it must be player 1's turn.

We are still on our way to prove that s_1 is dominated in the reduced game $D(N)$. Consider the alternative strategy $s_1' \in S_1$ that is equal to s_1 outside of the subgame $G(x_{h^*-1})$, and that is equal to some maximin strategy (guaranteeing player 1 a utility of 0) in $G(x_{h^*-1})$. We will show that s_1' dominates s_1 in $D(N)$.

We prove first that s_1' is at least as good as s_1 against any strategy s_2' available in $D(N)$ for player 2. Assume to the contrary that there is some strategy s_2' in $D(N)$ such that

$$u_1(s_1', s_2') < u_1(s_1, s_2'). \tag{2}$$

Then the node x_{h^*-1} is reached by both the profiles (s_1', s_2') and (s_1, s_2'). (If none of the two profiles reached the subgame $G(x_{h^*-1})$, then utility levels could not differ. Moreover, since s_1 and s_1' are equal outside of $G(x_{h^*-1})$, one profile reaches the subgame if and only if the other does so.) Thus, as s_1' induces a maximin strategy in $G(x_{h^*-1})$, and as $v_1(x_{h^*-1}) = 0$, we get $u_1(s_1', s_2') \geq 0$. From (2), it follows that $u_1(s_1, s_2') = 1$.

Now, the profile (s_1, s_2') reaches even the node x_{h^*}. (The profile (s_1, s_2') reaches x_{h^*-1}, and at the node x_{h^*-1} player 1 is called to play, as was shown previously.) We show that in this case, s_2' is dominated in N. Consider the alternative strategy $s_2'' \in S_2$ which is equal to s_2' outside of the subgame $G(x_{h^*})$ and is equal to some winning strategy in $G(x_{h^*})$. (Recall that $v_2(x_{h^*}) = 1$.) The strategy s_2'' is at least as good as s_2' for all possible strategies for player 1 (by an argument similar to the one used above to show that $\tilde{s}_1$ dominates s_1), and strictly better against s_1 since

$$u_2(s_1, s_2') = -u_1(s_1, s_2') = -1 < 1 = u_1(s_1, s_2'').$$

(Note that (s_1, s_2'') reaches $G(x_{h^*})$, since (s_1, s_2'') is equal to (s_1, s_2) outside of the subgame $G(x_{h^*})$.) Hence, s_2' is dominated in N, and therefore there does not exist a strategy s_2' in $D(N)$ satisfying (2).

We show now that s_1' is sometimes strictly better than s_1 in $D(N)$. This is the case when player 2 chooses s_2. To see this note first that (s_1', s_2) reaches

	L	C	R
T	1,-1	0,0	-1,1
M	0,0	0,0	0,0
B	-1,1	0,0	1,-1

FIG. 1. A strictly competitive game with three outcomes in which no strategy is dominated.

the subgame $G(x_{h^*-1})$. (This is so because (s_1', s_2) is equal to (s_1, s_2) outside of $G(x_{h^*})$, hence a fortiori outside of $G(x_{h^*-1})$.) Hence, as $v_1(x_{h^*-1}) = 0$,

$$u_1(s_1', s_2) \geq 0 > -1 = u_1(s_1, s_2).$$

In sum, this shows that s_1 is dominated in $D(N)$ by s_1', and concludes the analysis of Case B, so that the theorem is proved. ■

The proof makes extensive use of the perfect-information assumption. Once this assumption is dropped, the result ceases to hold. To see this, consider the normal-form game N_1 depicted in Fig. 1. This game satisfies all assumptions of Theorem 1 except that it is of imperfect information. And, there is no dominated strategy in N_1.

4. CONCLUSION

We have shown that chess-like games can be solved by applying only two rounds of elimination of dominated strategies.[12]

Our result raises the following, as we find, puzzling issue. The standard way for determining the value of a finite zero-sum game of perfect information is by backward reasoning—beginning with the decisions to be made at the end of the game, and ending with the decisions to be made at the beginning of the game. In terms of rationality, this seems to require that players hold mutual beliefs (or knowledge) of rationality (in the sense that "White believes that Black believes that White believes and so on") of a degree equal to the (potentially huge!) maximal number of moves of the game.[13] In contrast, when players determine the value of the extensive-form

[12] In terms of computational complexity, it should make a serious difference whether a game is analyzed one way or the other. In the traditional backward induction procedure, every node is visited once, and in the elimination procedure, every pair of strategies must be checked for dominance, twice, and for both players.

[13] Aumann (1995) writes that backward induction presupposes "that all players know that all are rational, all know this, all know *this*, and so on [...] at least, for a number of levels no less than the maximum duration of the game." Balkenborg and Winter (1997) formalize and prove this result for *generic* perfect information games.

game by iterated dominance in the strategic form, then the necessary level of mutual beliefs of rationality seems to be only two![14]

We conjecture that the following more general statement is true: Any strictly competitive, finite game of perfect information with n outcomes can be solved by applying $n - 1$ rounds of elimination of dominated strategies.[15]

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[14]This observation may be related to well-known puzzles associated with the iterated elimination of weakly dominated strategies, cf., e.g., Samuelson (1992), Börgers and Samuelson (1992), Ewerhart (1998), and the discussion in Gul (1996).

[15]In fact, any such game is dominance solvable (see Moulin, 1979, Prop. 2, and Gretlein, 1982). However, the existing proof is based on a backward induction argument, and therefore the resulting bound for the number of iterations is not lower than the maximal number of moves in the game.